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THE REFRACTION OF A PURE SHEAR SHOCK WAVE INTO AN ELASTIC-PLASTIC HALF-SPACE*

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Regularities in the propagation of an unloading shock wave are investigated as a development of the results of solving the problem of the refraction of a pure shear plane elastic wave of arbitrary profile into an ideal elastic-plastic half-space. An analytic solution is constructed for the problem of the refraction of a wave having the shape of a step of finite length in both the active plastic loading domain and in the unloading zone.

1. During pure shear wave propagation the medium is under antiplane deformation conditions, the displacement velocity vector w is directed along the x_3 axis, and only depends on the variables x_1, x_2 and the time t, and only the stresses $\tau_1 = \sigma_{13}(x_1, x_2, t)$ and $\tau_2 = \sigma_{23}(x_1, x_2, t)$ are non-zero. The equations of the dynamics of an ideal elastic-plastic body are written down in /1/ for this case. Henceforth we shall confine ourselves to investigating the selfsimilar solutions of the equations of the dynamics of ideal elastic-plastic media which depend on two variables $x = x_1 - ct$ and $y = x_2$. The equations of the characteristics of the system of equations of motion and the relationships along the characteristics here have the following form in the active plastic loading domain /1/:

$$dy\,(\mathrm{M} \pm \cos\theta) = \mp \sin\theta\, dx, \quad k\theta \pm \rho a w = \mathrm{const} \tag{1.1}$$

Here k is the yield point, ρ is the density, μ is the shear modulus, $a = \sqrt{\mu/\rho}$ is the velocity of propagation of transverse elastic waves, M is the Mach number and θ is a quantity such that $\tau_1 = k\sin\theta$, $\tau_2 = k\cos\theta$ take only the upper or lower signs, respectively.

The general integral /1/

$$c\tau_1 + \mu w = f(y) \tag{1.2}$$

holds in the elastic domain and in the unloading zone, while the equations of the characteristics of the system of equations of motion and the relationships along the characteristics have the form /1/

$$x \pm \sqrt{\mathrm{M}^2 - 1}\, y = \mathrm{const}, \quad \mu\sqrt{\mathrm{M}^2 - 1}\, w \mp c\tau_2 = \mathrm{const} \tag{1.3}$$

Let a pure shear plane wave OA (Fig.1) be incident from the elastic half-space $y < 0$ with parameters $\mu_1, \rho_1, a_1 = \sqrt{\mu_1/\rho_1}$ on the interfacial boundary $y = 0$ with the elastic-plastic half-space $y > 0$ which is characterized by the parameters $\mu_2, \rho_2, a_2 = \sqrt{\mu_2/\rho_2}, k$ whereupon a reflected wave OB and a refracted wave OC is formed as a result of its interaction with the interfacial boundary. The material in front of the refracted wave front OC is at rest and there are no initial stresses therein. Complete contact between the elastic and elastic-plastic half-spaces is assumed on the interfacial boundary, i.e., the normal stress τ_2 and the displacement velocity w are continuous on the interfacial boundary $y = 0$, whence we have /1/

$$\begin{gathered} w(x) = w_1(-x\sin\varphi_1) + w_2(-x\sin\varphi_1) \\ \tau_2(x) = \mu^{-1}\operatorname{ctg}\varphi_1\,(w_2(-x\sin\varphi_1) - w_1(-x\sin\varphi_1)), \quad \mu = \mu_2/\mu_1 \end{gathered} \tag{1.4}$$

Here $w(x)$ is the displacement velocity, $\tau_2(x)$ is the stress on the interfacial boundary in the elastic-plastic half-space, $w_1(-x\sin\varphi_1)$ is a function giving the incident wave profile and intensity (considered known according to the formulation of the problem), $w_2(-x\sin\varphi_1)$ is the reflected wave intensity, and φ_1 is the angle of incidence.

System (1.4) is written in dimensionless variables that will also be used later. These dimensionless variables are chosen as follows

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$$\bar{x}=\frac{x}{l},\quad \bar{y}=\frac{y}{l},\quad \bar{\tau}_1=\frac{\tau_1}{k},\quad \bar{\tau}_2=\frac{\tau_2}{k},\quad \bar{w}=\frac{w}{w^*},\quad \bar{w}_1=\frac{w_1}{w^*},$$
$$\bar{w}_2=\frac{w_2}{w^*},\quad w^*=\frac{ck}{\mu_2}$$

l is the characteristic length, and w^* is the characteristic velocity. For simplicity, we shall henceforth discard the bars above the letters.

A solution of the above problem was constructed in /1/ for an arbitrary smooth incident wave profile, i.e., for the case when the function $w_1(-x\sin\varphi_1)$ has no discontinuities. The following waves propagate successively in the elastic-plastic half-space: an elastic load wave, a plastic loading wave, and an unloading wave (UW), which is a wave of weak discontinuity for the case of an incident wave of smooth profile. The construction of an UW of weak discontinuity reduces to solving systems of functional equations of complex structure /2/, which constrains the possibility of applying analytic investigation methods. An algorithm for the numerical construction of the UW of weak discontinuity has been described /3/ for the problem under consideration.

2. We consider the propagation of a shock unloading wave within the framework of the problem in question. Later the shockwave will be understood to be an isolated surface moving in space on which the stress and displacement velocity undergo a discontinuity.

The equation of conservation of momentum should be satisfied on the shock UW /4/, which in the case under consideration has the form

$$[\tau_1]\nu_1+[\tau_2]\nu_2+c_1a_2^{-1}M\,[w]=0 \tag{2.1}$$

Here ν_1, ν_2 is the projection of the normal vector on the x, y axes, c_1 is the UW velocity, and $[\tau]=\tau^+-\tau^-, \tau^+, \tau^-$ are the limit values of τ on the UW from the plastic loading and unloading domains, respectively.

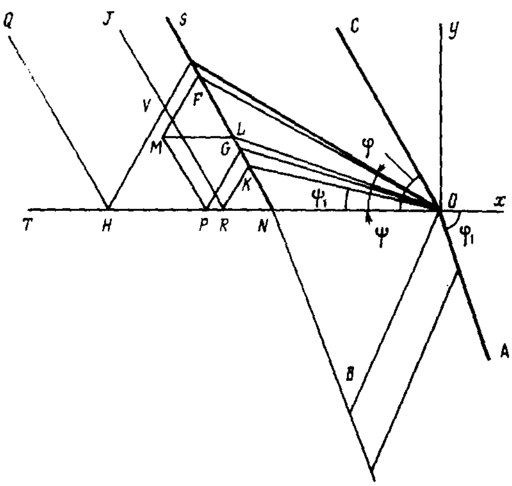

Fig.1

It has been shown /5/ that plastic deformation jumps occur only in singular cases, consequently, the plastic deformations are continuous on the UW front. Using Hooke's law, and the kinematic and geometric compatibility conditions /5/, we obtain for the stress jumps

$$[\tau_i]=-cc_1^{-1}\nu_i\,[w],\quad i=1,2 \tag{2.2}$$

Substituting (2.2) into (2.1) and taking into account that $\nu_1=\sin\varphi,\ \nu_2=\cos\varphi$, we obtain

$$[w]\,(\mu_2-\rho_2c_1^2)=0 \tag{2.3}$$

It follows from (2.3) that the shock UW velocity of propagation equals the elastic wave velocity of propagation $c_1=a_2=\sqrt{\mu_2/\rho_2}$ and the relationships on the line of discontinuity (2.2) take the form

$$[\tau_1]=-[w],\quad [\tau_2]=-\sqrt{M^2-1}\,[w],\quad c=a_2/\sin\varphi,\quad M=ca_2^{-1} \tag{2.4}$$

Here φ is the angle of refraction such that

$$a_1\sin\varphi=a_2\sin\varphi_1 \tag{2.5}$$

If the material on the UW front is in the elastic state, then a constraint on the magnitude of the velocity jump follows from the yield condition and (2.4)

$$2[w](\tau_1^+ + \sqrt{M^2-1}\,\tau_2^+) + M^2[w]^2 \leqslant 0 \tag{2.6}$$

The quantity $[w]$ remains undetermined in (2.6).

We will assume that the material behind the UW front is in the elastic state, i.e., condition (2.6) is satisfied. The shock UW has the form

$$x + \sqrt{M^2-1}\,y = x_N \tag{2.7}$$

Here x_N is a point on the interfacial boundary at which the UW starts to propagate.

The second relationship of (1.3) with the upper sign holds on the line (2.7), and we can write it in the form

$$\sqrt{M^2-1}\,w^- - \tau_2^- = \sqrt{M^2-1}\,w_N^- - \tau_{2N}^- \tag{2.8}$$

Here and henceforth, the letter subscript on the quantities $x, \tau_1, \tau_2, w, \theta$ (for instance, the subscript N in (2.8)) means that the corresponding quantity is evaluated at the point denoted by this letter in Fig.1. The quantities w^- and τ_2^- are evaluated at an arbitrary point of the line (2.7) in the unloading zone.

We have from relationships (2.4) for an arbitrary point of the UW, including also for the point N

$$\sqrt{M^2-1}\,w^- + \tau_2^- = \sqrt{M^2-1}\,w^+ + \tau_2^+ \tag{2.9}$$

The boundary condition on the interfacial boundary (1.6) can be written for the point N in the unloading zone in the form

$$2w_1(-x\sin\varphi_1) = w_N^- - \mu\,\mathrm{tg}\,\varphi_1\tau_{2N}^- \tag{2.10}$$

The system of Eqs.(2.8)-(2.10) enables us to determine the quantities τ_2^- and w^- on the line (2.7) in the unloading zone if the solution in the active plastic loading domain is known ahead of the UW front.

Initially we will assume that the stress tensor components τ_1, τ_2 and the displacement velocity w undergo a discontinuity at the point N. Then, writing relationships (2.9) for the point N and solving it in combination with (2.10), the quantities τ_{2N}^- and w_N^- can be determined and the quantities τ_2^- w^+ are then determined from (2.8) and (2.9) and at an arbitrary UW point

$$w^- = A + B^+/\sqrt{M^2-1}, \quad \tau_2^- = -A\sqrt{M^2-1} + B^- \tag{2.11}$$
$$A = \frac{2w_1(-x_N^-\sin\varphi_1)\,\Delta}{\Delta+\cos\varphi}, \quad B^\pm = \frac{F(x^+) \pm \Delta_1 F(x_N^+)}{2}$$
$$\Delta = \mu^{-1}\sin\varphi\,\mathrm{ctg}\,\varphi_1, \quad \Delta_1 = (\cos\varphi - \Delta)/(\Delta+\cos\varphi)$$
$$F(x^+) = \sqrt{M^2-1}\,w^+ + \tau_2^+$$

The relations (2.11) obtained enable us to write

$$\tau_1^- = \tau_1^+ - A + C/\sqrt{M^2-1} \tag{2.12}$$
$$C = {}^1/_2\,(\sqrt{M^2-1}\,w^+ - \tau_2^+ - \Delta_1 F(x_N^+))$$

after using (2.4).

The solution constructed holds in the case when condition (2.6) is satisfied, which can be represented by using relationship (2.11), in the form

$$2(\tau_1^+ + \tau_2^+\sqrt{M^2-1}) + M^2(A + C/\sqrt{M^2-1}) \leqslant 0 \tag{2.13}$$

Condition (2.13) is written for $[w] > 0$. If $[w] < 0$ then the symbol $\leqslant$ in (2.13) should be replaced by the symbol $\geqslant$.

Let us examine another possible case when the stress and displacement velocity are continuous at the point N but jumps τ_1, τ_2, w are later formed on the shock UW. Then

$$w_N^+ = w_N^-, \quad \tau_{1N}^+ = \tau_{1N}^-, \quad \tau_{2N}^+ = \tau_{2N}$$

In this case the integral (2.8) and the second condition on the line of strong discontinuity (2.4) take the following form for an arbitrary UW point:

$$\sqrt{M^2-1}\,w^- - \tau_2^- = \sqrt{M^2-1}\,w_N^+ - \tau_{2N}^- \tag{2.14}$$
$$\sqrt{M^2-1}\,w^- + \tau_2^- = \sqrt{M^2-1}\,w^+ + \tau_2^+$$

Solving system (2.14), we obtain expressions for τ_2^- and w^- which are used to determine the jumps of w and τ_2 at an arbitrary UW point

$$[w]=\frac{w^+-w_N^+}{2}-\frac{\tau_2^+-\tau_{2N}^+}{2\sqrt{M^2-1}} \qquad (2.15)$$
$$[\tau_2]=\frac{\tau_2^+-\tau_{2N}^+}{2}-\frac{\sqrt{M^2-1}}{2}(w^+-w_N^+)$$

Because in the plastic loading domain $0\leqslant w^+\leqslant w_N^+$, $\tau_{2N}^+\leqslant\tau_2^+\leqslant 0$ /1/, it follows from the first relationship in (2.15) that $[w]<0$, and condition (2.6) takes the form

$$2(\tau_1^++\tau_2^+\sqrt{M^2-1})+M^2\left(\frac{w^+-w_N^+}{2}-\frac{\tau_2^+-\tau_{2N}^+}{2\sqrt{M^2-1}}\right)\geqslant 0 \qquad (2.16)$$

But it follows from the solution of the active plastic loading domain /1/ that there is a negative quantity on the left side of the inequality (2.16). this means that condition (2.16) can be satisfied only when its left side is zero, i.e., for $\tau_1^+=\tau_2^+=0$. This is impossible since the material ahead of the UW front is in the plastic state.

Therefore, when the stress tensor components and the displacement velocity are continuous at the point N the shock UW cannot propagate.

The displacement velocity jump turns out to be positive on the UW .

Indeed, it follows from (2.4) that

$$\tau_1^-=\tau_1^++[w],\quad \tau_2^-=\tau_2^++\sqrt{M^2-1}\,[w] \qquad (2.17)$$

Since the quantities $\tau_1^+\leqslant 0$ and $\tau_2^+\leqslant 0$ in the plastic domain /1/, it then follows from (2.17) that the quantities τ_1^- and τ_2^- for $[w]<0$ increase simultaneously in absolute value and hence $(\tau_1^-)^2+(\tau_2^-)^2>1$, which contradicts the plasticity condition.

We will henceforth confine ourselves to investigating only the case $[w]>0$ on the UW.

It follows from the first equation in (2.11) that the condition of no jumps in the stress and the displacement velocity at the point N has the form

$$2w_1(-x_N^-\sin\varphi_1)=w_N^+-\mu\operatorname{tg}\varphi_1\tau_{2N}^+$$

Hence, the jump in the stress and the displacement velocity at the point N of an elastic-plastic half-space is non-zero (and therefore, the UW is a shock) in two cases: firstly, when the function $w_1(-x\sin\varphi_1)$ undergoes a discontinuity at the point N (for instance, if the incident wave is in the shape of a step), and secondly when satisfaction of the boundary condition on the interfacial boundary is violated at the point N.

3. We will use the relationships obtained to investigate the refraction of shear waves travelling from an elastic into an elastic-plastic half-space when the incident wave profile has the form of a step of finite length l, i.e.,

$$w_1(-x\sin\varphi_1)=\begin{cases} w_0=\text{const}, & x_N\leqslant x<0\\ 0, & x<x_N\end{cases}$$

The solution obtained in /1/ for the case of the incidence of a wave of arbitrary profile enables the solution of the problem to be constructed quite easily in the elastic domain and in the plastic loading domain behind the front of the refracted wave OC (Fig.1).

If the incident wave intensity w_0 satisfies the inequality

$$2\,|\,w_0\,|\leqslant\Delta_2/M,\ \Delta_2=1+\mu\operatorname{tg}\varphi_1/\operatorname{tg}\varphi \qquad (3.1)$$

then the material in the elastic-plastic half-space is in the elastic space. The equality symbol in conditions (3.1) corresponds to the condition of the material of the elastic-plastic half-space reaching the yield point. The displacement velocity w and the stresses τ_1, τ_2 behind the wave front OC are constant

$$w=2w_0/\Delta_2,\quad \tau_1=-2w_0/\Delta_2,\quad \tau_2=-2\sqrt{M^2-1}\,w_0/\Delta_2$$

If inequality (3.1) is not satisfied then the material in the elastic-plastic half-space is deformed plastically. In this case the point O is the source of a wave packet. The shock OC propagates at the velocity of the elastic waves. The stresses and displacement velocity between the characteristics OC and OE are constant. The yield point is reached in this zone but the plastic strain rates are zero. In the domain COE

$$w=\sin\varphi,\ \tau_1=-\sin\varphi,\ \tau_2=-\cos\varphi \qquad (3.2)$$

The characteristic of the negative direction of the family (1.1) is inclined at an angle ψ to the Ox axis, for which we have

$$\operatorname{tg}\psi = -\sin\varphi/(M+\cos\varphi)$$

Later the slope of the characteristics of the negative direction of the family (1.1) issuing from the point O diminishes to the magnitude ψ_1 such that

$$\operatorname{tg}\psi_1 = \sin\theta_1/(M-\cos\theta_1) \tag{3.3}$$

The quantity θ_1 in relationships (3.3) satisfies the boundary condition on the interfacial boundary (1.4), which we write in the form

$$2\mu^{-1}\operatorname{ctg}\varphi_1 w_0 = \Delta(1+\pi+\varphi-\theta_1) - \cos\theta_1 \tag{3.4}$$

Plastic deformation of the material occurs behind the wave front OE. The stresses and the displacement velocity at points between the characteristics OE and OK are determined as follows

$$\tau_1 = \sin\theta,\ \tau_2 = \cos\theta,\ w = \sin\varphi(1+\pi+\varphi-\theta) \tag{3.5}$$

Here θ is the root of the equation governing the location of the appropriate characteristic of the family (1.1) passing through the point O

$$y = x\sin\theta/(M-\cos\theta) \tag{3.6}$$

where x, y are the coordinates of the point at which the stresses and displacement velocity are evaluated.

The quantities τ_1, τ_2, and w between the characteristic OK and the line ON (Fig.1) take constant values and are evaluated by means of (3.5) for $\theta=\theta_1$.

When the incident wave intensity satisfies the inequality

$$w_0| \geqslant \sqrt{\mu\rho}\,(2\sin\varphi\cos\varphi_1)^{-1}(\Delta(1+\varphi)+1) \tag{3.7}$$

a slip zone is formed on the interfacial boundary. In this case the characteristic OK becomes parallel to the Ox axis. The line OK is a stationary line of discontinuity on which the displacement velocity undergoes a discontinuity, while the stress τ_2 is continuous across the line OK /1/. The mechanism of formation and the physical interpretation of the slip zone were discussed in detail in /1, 3/. In the formulation udner consideration the solution in the loading domain is obtained from the solution constructed in the limit case in /1/ when the characteristics of the family (1.1) issue from one point and the elastic zone is represented in the wave pattern (Fig.1) just by the domain of the constant state of stress.

In the UW case under consideration, NS which propagates from the point N (Fig.1) will be a shock since the function $w_1(x-\sin\varphi_1)$ undergoes a discontinuity at the point N. Therefore, the UW propagates at the elastic wave velocity and its location in the xOy plane is determined by (2.7).

Because the UW equation is known, an analytic solution is constructed successfully behind the wave front NS by the method of characteristics.

We proceed as follows to determine the quantities τ_2 and w at an arbitrary point M of the unloading domain. Characteristics MF and MP (Fig.1) of the family (1.3) are drawn through the point M. The characteristic PG of the family (1.3) (with the lower sign) is drawn from the point of intersection of the characteristic MP with the interfacial boundary to the intersection with the UW, and using (2.11) a relationship that holds along PG can be represented in the form

$$\sqrt{M^2-1}\,w \mp \tau_2 = \sqrt{M^2-1}\,w_G^+ + \tau_{2G}^+ \tag{3.8}$$

Moreover, the boundary condition (1.4) is satisfied at the point P and takes the following form in the unloading zone:

$$w = \mu\operatorname{tg}\varphi_1\tau_2 \tag{3.9}$$

Solving (3.8) and (3.9) simultaneously, we determine the quantities w and τ_2 at an arbitrary point P of the interfacial boundary

$$\tau_{2p} = \Delta F(x_G^+)/(\Delta+\cos\varphi),\quad w_p = \mu\operatorname{tg}\varphi_1\tau_{2p} \tag{3.10}$$

This enables us to evaluate the constant in the relationship along the characteristic MP

$$\sqrt{M^2-1}\,w - \tau_2 = \Delta_1 F(x_G^+) \tag{3.11}$$

Relationship (1.3) (with the lower sign) holds along the characteristic, and using (2.11) can be represented in the form

$$\sqrt{M^2-1}\,w+\tau_2=\sqrt{M^2-1}\,w_F^+ +\tau_{2F}^+ \qquad (3.12)$$

Solving (3.11) and (3.12) jointly, we determine the stress τ_2 and the displacement velocity w at an arbitrary point M of the unloading zone

$$\tau_{2M}={}^1/_2(F(x_F^+)-\Delta_1F(x_G^+)),\quad w_M={}^1/_2\operatorname{tg}\varphi\,(F(x_F^+)+\Delta_1F(x_G^+)) \qquad (3.13)$$

Let us note certain singularities of the solution constructed. The quantities τ_2 and w along a segment NK in the unloading zone are determined by (2.11) and since $\tau_2^+=\tau_{2N}^+$, $w^+=w_N^+$ for any point of the segment NK, it follows from (3.13) that the quantities τ_2 and w take constant values in the domain RKN. It hence follows that along segments of characteristics of the family (1.3) (with the lower sign) enclosed between the characteristics NS and JR (Fig.1), the quantities τ_2 and w remain constant.

The material has no plastic deformations in the domain $SEOC$. The stresses and displacement velocity are constant and determined by (3.2), consequently, the quantities τ_2 and w do not change along the line SE (characteristics of the family (1.3) with the upper sign). Therefore, τ_2 and w in the domain $JSEV$ take constant values (since the constants in the relationships along the characteristics of both families are the same for all characteristics of each family).

We obtain for points of the domain $JSEV$ from the relationships (3.13)

$$w={}^1/_2\operatorname{tg}\varphi\Delta_1F(x_N^+),\quad \tau_2={}^1/_2\Delta_1F(x_N^+) \qquad (3.14)$$

It was taken into account in deriving (3.14) that

$$\sqrt{M^2-1}\,w+\tau_2=0 \qquad (3.15)$$

along characteristics of the family (1.3) (with the lower sign) intersecting the line SE.

Solving (3.9) and (3.15) simultaneously, we find $\tau_{2H}=w_H=0$, i.e., in the rest zone TQH there is no stress τ_2.

We have the relationship (1.2) to determine the stress τ_1, and which has the following form along the line $y=\text{const}$ in the unloading zone:

$$y=\text{const},\ \tau_1+w=\text{const} \qquad (3.16)$$

We obtain from conditions on the line of strong discontinuity (2.4) for the quantity τ_1

$$\tau_1^-+w^-=\tau_1^++w^+ \qquad (3.17)$$

where the quantities $\tau_1^-,\tau_1^+,w^-,w^+$ are evaluated at the point of intersection of the line $y=\text{const}$ with the UW.

It follows from (3.16) and (3.17) that in the unloading domain

$$y=\text{const},\ \tau_1+w=\tau_1^++w^+ \qquad (3.18)$$

Because the quantity w was determined above, τ_1^+ and w^+ are evaluated by means of (3.15) in the plastic loading domain, and we determine the stress τ_1 at an arbitrary point M of the unloading zone (Fig.1) from the relationships (3.19)

$$\tau_{1M}=\tau_{1L}^++w_L^+-{}^1/_2\operatorname{tg}\varphi\Delta_1(F(x_F^+)+F(x_G^+)) \qquad (3.19)$$

It follows from (3.19) that after rest has built up behind the UW front, the stress τ_1 differs from zero in a layer of depth h of the elastic plastic half-space adjoining the interfacial boundary. The magnitude of the residual stress equals

$$\tau_1^*=\tau_{1L}^++w_L^+=\sin\theta_L+\sin\varphi\,(1+\pi+\varphi-\theta_L) \qquad (3.20)$$

Analysis of (3.20) shows that the residual stresses are positive, the quantity τ_1^* takes the maximum value on the interfacial boundary and falls to zero as the coordinate y increases. The residual stress distribution depends on the incident wave intensity w_0, and the parameters φ_1,μ,ρ but is independent of the duration of the action (i.e., of the length of the incident wave l).

We note that the greatest value of τ_1^* is on the interfacial boundary when a slip zone occurs on it and this value is independent of w_0 for w_0 satisfying condition (3.7).

The dependence of the residual stresses on the incident wave intensity w_0 is represented in Fig.2 for the following parameter values:

$$\varphi_1=80^\circ,\ \mu=0.5,\ \rho=0.6,\ l=10$$

Using theorems from geometry we obtain the layer depth from the construction in Fig.1, at which $\tau_1^*\neq 0$

$$h=\sin\varphi\sin 2\varphi_1\sin\psi/[\sin\varphi_1\sin(\varphi-\psi)] \qquad (3.21)$$

Using (3.3) and (2.5), expression (3.21) with the condition $\sin\varphi_1\neq 0,\cos\varphi\neq 0$ can be converted to the form

$$h = 2\sin^2\varphi_1 \cos\varphi_1 \mu l/\rho$$

The solution constructed enables us to determine the reflected wave intensity $w_2(-x\sin\varphi_1)$. Using (1.4) and (3.5), we determine the quantity $w_2(-x\sin\varphi_1)$ for points of the interfacial boundary ahead of the UW front, i.e., for $x \in [x_N, 0]$

$$w_2(-x\sin\varphi_1) = w_0 - \mu \operatorname{tg}\varphi_1 \cos\theta_1$$

It follows from (1.4) that behind the UW front

$$w_2(-x\sin\varphi_1) = w(x)$$

where $w(x)$ is evaluated by means of (3.10).

Qualitative features of the solution are illustrated by the graphs in Fig. 3 for the following parameter values:

$$\varphi_1 = 80^\circ,\ \mu = 0.5,\ \rho = 0.6,\ w_0 = 1.8,\ l = 15$$

Graphs are presented of the change in the stresses τ_1 and τ_2 (solid lines *1* and *2*) behind the refracted wave front OC for $y = 4$. An analogous dependence for the velocity w is represented by the dashed line *1*. A change in the reflected wave intensity w_2 on the interfacial boundary is shown by the dashed line *2*.

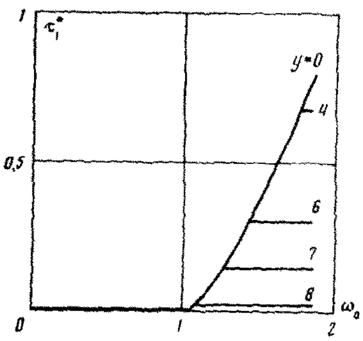

Fig.2

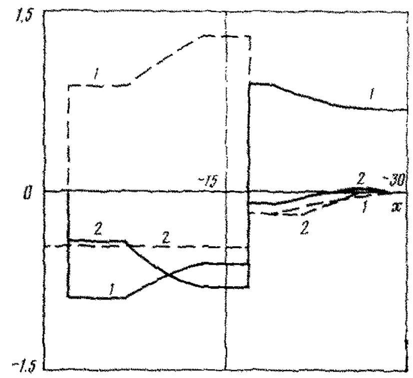

Fig.3

4. The solution constructed holds when the material behind the front is in the elastic state, i.e., the stresses calculated by means of (3.13) and (3.19), satisfy the inequality

$$\tau_1^2 + \tau_2^2 \leqslant 1 \tag{4.1}$$

To simplify the subsequent writing, we introduce the notation

$$z = 2[\sin\theta_L + \sin\varphi(1 + \pi + \varphi - \theta_L)],\quad z_1 = \cos\theta_F + \cos\varphi(1 + \pi + \varphi - \theta_F) \tag{4.2}$$

$$z_2 = \cos\theta_G + \cos\varphi(1 + \pi + \varphi - \theta_G)$$

We then obtain from (4.2), (3.13), and (3.20)

$$F_1 = F_1(z, z_1, z_2) = [z - \operatorname{tg}\varphi(z_1 + \Delta_1 z_2)]^2 + [z_1 - \Delta_1 z_2]^2 = 4(\tau_1^2 + \tau_2^2) \tag{4.3}$$

Let us determine the extrema of the function F_1. The determinant of the system of equations obtained by equating the partial derivatives of the function F_1 with respect to z, z_1, z_2 to zero equals zero. Therefore, this system has an infinite set of solutions of the form

$$z_1^\circ = \Delta_1 z_2,\quad z^\bullet = \Delta_1 z_2 \operatorname{tg}\varphi \tag{4.4}$$

Substitution of the solutions (4.4) into (4.3) yields $F_1(q^\circ, z_1^\circ, z_2) = 0$. The function F_1 therefore takes the greatest value on the boundary of the domain of definition of its arguments.

We will investigate the behaviour of the function F_1 on the characteristics of the family (1.3) (with the lower sign). In this case the search for the extrema of the function F_1 reduces to the problem of determining the conditional extremum of the function F_1 under the condition

$$z_1 = \text{const} \tag{4.5}$$

We obtain from (4.3)-(4.5) and the necessary condition for the existence of a conditional extremum that the function F_1 can have a conditional extremum only at the points $z_2^* = \Delta_1 z_1$, $z^* = \operatorname{tg}\varphi\, z_1$. But $F_1(z^*, z_1, z_2^*) = 0$. Consequently, the function F_1 takes its greatest value on characteristics of the family (1.3) at points of intersection of these later with the UW or with the interfacial boundary.

The elastic solution holds along the characteristic NS of the family (1.3) (with the upper sign) Fig.1) in the unloading domain when condition (2.13) is satisfied, and we can write it in the form

$$F_2(\theta_F, \theta_N) = 2(\sin\theta_F + \sqrt{M^2-1}\cos\theta_F) + {}^1/_2 M(1+\pi+\varphi-\theta_F - \Delta_1(1+\pi+\varphi-\theta_N)) - {}^1/_2(M^2/\sqrt{M^2-1})(\cos\theta_F + \Delta_1\cos\theta_N) \leqslant 0 \quad (4.6)$$

(F is an arbitrary point of the characteristic NS). Differentiating the function $F_2(\theta_F, \theta_N)$ along the characteristic NS we obtain

$$\frac{\partial F_2}{\partial \bar{m}} = \frac{\partial F_2}{\partial \theta_F}\left(\frac{\partial \theta_F}{\partial x}\cos\alpha_1 + \frac{\partial \theta_F}{\partial y}\cos\alpha_2\right) + \frac{\partial F_2}{\partial \theta_N}\left(\frac{\partial \theta_N}{\partial x}\cos\alpha_1 + \frac{\partial \theta_N}{\partial y}\cos\alpha_2\right) = \left(-\frac{2\sin(\theta_F-\varphi)}{\sin\varphi} - \frac{M}{2}\left(1 - \frac{M}{\sqrt{M^2-1}}\sin\theta_F\right)\right)\left(-\frac{\partial\theta_F}{\partial x}\sin\varphi + \frac{\partial\theta_F}{\partial y}\cos\varphi\right) \quad (4.7)$$

Here $\bar{m}$ is the unit vector giving the direction of the characteristics NS with the coordinates $m_i = \cos\alpha_i$, α_i are the angles between the vector $\bar{m}$ and the positive directions of the coordinate axes. It was taken into acccount in deriving (4.7) that the quantity θ_N in (4.6) remains constant for any point of the characteristic NS.

We have $\theta_F \in [\pi, \pi+\varphi]$, $\partial\theta_F/\partial x \leqslant 0$, $\partial\theta_F/\partial y \geqslant 0$ from the solution constructed earlier in the plastic loading domain. Consequently, we obtain from (4.7) that $\partial F_2/\partial\bar{m} \leqslant 0$. Therefore, the function $F_2(\theta_F, \theta_N)$ takes the greatest value at the point N. But since the function $F_1(z, z_1, z_2)$ on any characteristic of the family (1.3) (with the lower sign) takes the greatest value at the point of intersection with either the interfacial boundary or with the UW, we then find that codition (4.1) can be violated first at points of the interfacial boundary. Therefore, by using (3.10) and (3.19) we conclude that the solution constructed holds when the following inequality is satisfied:

$$[\tau_{1N}^+ + w_N{}^\tau - \operatorname{tg}\varphi\Delta_1 F(x_G)]^2 + [F(x_G)\Delta_1/(\Delta_1+\cos\varphi)]^2 \leqslant 1$$

Otherwise, secondary plastic flows occur behind the UW front and the construction of the solution then requires a separate investigation.

REFERENCES

1. BYKOVTSEV A.G., Plane-polarized wave refraction on the boundary between elastic and elastic-plastic half-spaces, PMM, 49, 2, 1985.
2. RAKHMATULIN KH.A. and DEM'YANOV YU.A., Strength Under Intense Short-range Loads. Fizmatgiz, Moscow, 1961.
3. BYKOVTSEV A.G., On shear wave refraction in a non-linearly elastic and elastic-plasic half-space, PMM 50, 3, 1986.
4. SEDOV L.I., Mechanics of a Continuous Medium, 1, Nauka, Moscow, 1970.
5. BYKOVTSEV G.I. and KRETOVA L.D., On shockwave propagation in elastic-plastic media, PMM, 36, 1, 1972.

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